

ON THE UPPER SEMI-CONTINUITY OF HSL NUMBERS

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ABSTRACT. Let B be an affine Cohen-Macaulay algebra over a field of characteristic p . For every prime ideal $\mathfrak{p} \subset B$, let $H_{\mathfrak{p}}$ be $H_{\mathfrak{p}B_{\mathfrak{p}}}^{\dim B_{\mathfrak{p}}}(\widehat{B_{\mathfrak{p}}})$. Each such $H_{\mathfrak{p}}$ is an Artinian module endowed with a natural Frobenius map Θ and if $\text{Nil}(H_{\mathfrak{p}})$ denotes the set of all elements in $H_{\mathfrak{p}}$ killed by some power of Θ then a theorem by Hartshorne-Speiser and Lyubeznik shows that there exists an $e \geq 0$ such that $\Theta^e \text{Nil}(H_{\mathfrak{p}}) = 0$. The smallest such e is the HSL-number of $H_{\mathfrak{p}}$ which we denote $\text{HSL}(H_{\mathfrak{p}})$.

The main theorem in this paper shows that for all $e > 0$, the sets $\{\mathfrak{p} \in \text{Spec } B \mid \text{HSL}(H_{\mathfrak{p}}) < e\}$ are Zariski open, hence HSL is upper semi-continuous.

This extends [H, Prop. 4.8] where M. Hashimoto proves that the F -injective locus of an F -finite Cohen-Macaulay ring is a Zariski open set. Note that our result does not assume F -finiteness.

1. INTRODUCTION

Throughout this paper every ring is assumed to be Noetherian, commutative, associative, with identity, and of characteristic p . Furthermore, we assume the reader has some prior knowledge of local cohomology, such as can be found in [B-S, Chapter 3].

If R is such a ring then for any positive integer e we define the e^{th} -iterated Frobenius endomorphism $T^e: R \rightarrow R$ to be the map $r \mapsto r^{p^e}$. For any R -module M , we can define $F_*^e M$ to be the Abelian group M with R -module structure given by $r \cdot m = T^e(r)m = r^{p^e}m$ for all $r \in R$ and $m \in M$. We can also define a functor, the e^{th} -Frobenius functor from R -modules to R -modules as follows. For any R -module M , we consider the $F_*^e R$ -module $F_*^e R \otimes_R M$ and after identifying the rings R and $F_*^e R$, we may regard $F_*^e R \otimes_R M$ as an R -module and denote it $F_R^e(M)$ or just $F^e(M)$ when R is understood. The functor $F_R^e(-)$ is exact when R is regular, cf. [B-H, Corollary 8.2.8], and for any matrix C with entries in R , $F_R^e(\text{Coker } C)$ is the cokernel of the matrix $C^{[p^e]}$ obtained from C by raising its entries to the p^e th power.

The main result of this paper concerns the study of HSL numbers which we now define. For any R -module M an additive map $\Theta : M \rightarrow M$ is a e^{th} -Frobenius map if it satisfies $\Theta(rm) = r^{p^e} \Theta(m)$ for all $r \in R$ and $m \in M$. Given such Θ we can define for $i \geq 0$ the R -submodule $M_i = \{m \in M \mid \Theta^i m = 0\}$. We define the *submodule of nilpotent elements in M* , denoted $\text{Nil}(M)$, to be $\cup_{i \geq 0} M_i$.

We have the following.

Theorem 1 (cf. Proposition 1.11 in [H-S] and Proposition 4.4 in [L]). *Assume (R, \mathfrak{m}) is a complete regular ring, M is an Artinian R -module and $\Theta : M \rightarrow M$ is a Frobenius map then the ascending sequence $\{M_i\}_{i \geq 0}$ above stabilises, i.e., there exists an $\eta \geq 0$ such that $\Theta^\eta \text{Nil}(M) = 0$.*

Definition 2. We define the *HSL number or index of nilpotency* of Θ on M , denoted $\text{HSL}(M)$, to be the smallest integer e at which $\Theta^e \text{Nil}(M) = 0$, or ∞ if no such i exists.

We can rephrase Theorem 1 by saying that under the hypothesis of the theorem, $\text{HSL}(M) < \infty$.

Our results are a generalisation of [H, Prop. 4.8] where it is proven that under certain hypothesis, the F -injective locus of a ring, which we introduce below, is open.

Recall that a *natural Frobenius map* acting on any R -module M induces a natural Frobenius map on $H_{\mathfrak{m}}^i(M)$, cf [K1, Sect.2].

Definition 3. A local ring (R, \mathfrak{m}) is *F -injective* if the natural Frobenius map $T : H_{\mathfrak{m}}^i(R) \rightarrow H_{\mathfrak{m}}^i(R)$ is injective for all i .

Remark 4. If (R, \mathfrak{m}) is a Cohen-Macaulay ring of dimension d and M is an R -module then the only non-trivial local cohomology module is the top local cohomology $H_{\mathfrak{m}}^d(M)$. Therefore, for such a module M to be F -injective is equivalent to $\text{HSL}(H_{\mathfrak{m}}^d(M)) = 0$.

We shall say that (R, \mathfrak{m}) is CMFI if it is Cohen-Macaulay and F -injective. A non-local ring R is CMFI if for each maximal ideal $\mathfrak{m} \subset R$ the localisation $R_{\mathfrak{m}}$ is CMFI. We define the *F -injective locus* of R to be:

$$\text{CMFI}(R) = \{\mathfrak{p} \in \text{Spec}(R) \mid R_{\mathfrak{p}} \text{ is CMFI}\}.$$

The structure of this paper is the following; in Section 2 we define the operator $I_e(-)$ and in the case of a polynomial ring A we show that it commutes with completions and localisations with respect to any multiplicatively closed subset of A . In Section 3 we are given a quotient

S of a local ring (R, \mathfrak{m}) and we give an explicit description of the R -module $\mathcal{F}_e(H_{\mathfrak{m}S}^d(S))$ consisting of all e^{th} -Frobenius maps acting on the top local cohomology module $H_{\mathfrak{m}S}^d(S)$. Using the fact that $\mathcal{F}_e(H_{\mathfrak{m}S}^d(S))$ is generated by one element which is the natural Frobenius map acting on $H_{\mathfrak{m}S}^d(S)$ (cf. [Ly-Sm, Example 3.7]) along with our result we obtain an explicit description of any Frobenius map acting on $H_{\mathfrak{m}S}^d(S)$. Once we know this, we prove the main theorem of this section whose aim is to give a formula to compute $\text{HSL}(H_{\mathfrak{m}S}^d(S))$; as a corollary we get a characterisation for S to be F -injective. The goal of Section 4 is to prove that the set $\mathcal{B}_e = \left\{ \mathfrak{p} \in \text{Spec}(A) \mid \text{HSL} \left(H_{\mathfrak{p}\widehat{B}_{\mathfrak{p}}}^{d(\mathfrak{p})}(\widehat{B}_{\mathfrak{p}}) \right) < e \right\}$, where B is a quotient of a polynomial ring A , is a Zariski open. Note that it follows that the F -injective locus of a quotient of a polynomial ring is Zariski open.

2. THE $I_e(-)$ OPERATOR

In this section we show that the $I_e(-)$ operator defined below commutes with localisations and completions. For any ideal I of R , we shall denote by $I^{[p^e]}$ the e^{th} -Frobenius power of I , i.e. the ideal generated by $\{a^{p^e} \mid a \in I\}$.

Definition 5. If R is a ring and $J \subseteq R$ an ideal of R we define $I_e(J)$ to be the smallest ideal B of R such that its e^{th} -Frobenius power $B^{[p^e]}$ contains J .

Remark 6. In general, such an ideal does not need to exist; however it does exist in polynomial rings and power series rings, cf [K1, Proposition 5.3].

Throughout Section 2 let A be a polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ and W be a multiplicatively closed subset of A and $J \subset A$ an ideal.

Lemma 7. *If $L \subseteq W^{-1}A$ is any ideal then $L^{[p^e]} \cap A = (L \cap A)^{[p^e]}$.*

Proof. $L^{[p^e]} \cap A$ and $(L \cap A)^{[p^e]}$ have the same generators. In fact, let $\frac{g_1}{1}, \dots, \frac{g_s}{1}$ generate L . Then $L^{[p^e]}$ is generated by $\frac{g_1^{p^e}}{1}, \dots, \frac{g_s^{p^e}}{1}$ and $L^{[p^e]} \cap A$ is consequently generated by $g_1^{p^e}, \dots, g_s^{p^e}$; on the other hand $L \cap A$ is generated by g_1, \dots, g_s therefore $(L \cap A)^{[p^e]}$ is generated by $g_1^{p^e}, \dots, g_s^{p^e}$. □

Lemma 8. *Let A be a polynomial ring and W any multiplicatively closed subset of A . Then for any ideal J of A and for any integer e , $I_e(W^{-1}J)$ exists and equals $W^{-1}I_e(W^{-1}J \cap A)$.*

Proof. Given any ideal $L \subseteq W^{-1}A$ such that $W^{-1}J \subseteq L^{[p^e]}$ we have that $W^{-1}I_e(W^{-1}J \cap A) \subseteq L$; in fact, $W^{-1}J \cap A \subseteq L^{[p^e]} \cap A = (L \cap A)^{[p^e]}$ and consequently $I_e(W^{-1}J \cap A) \subseteq L \cap A$ so $W^{-1}I_e(W^{-1}J \cap A) \subseteq W^{-1}(L \cap A) = L$. Hence $W^{-1}I_e(W^{-1}J \cap A)$ is contained in all the ideals L such that $W^{-1}J \subseteq L^{[p^e]}$. If we show that $W^{-1}J \subseteq (W^{-1}I_e(W^{-1}J \cap A))^{[p^e]}$ then $I_e(W^{-1}J)$ exists and equals $W^{-1}I_e(W^{-1}J \cap A)$. But since $W^{-1}J \cap A \subseteq I_e(W^{-1}J \cap A)^{[p^e]}$ then using Lemma 7 we obtain $W^{-1}J = W^{-1}(W^{-1}J \cap A) \subseteq W^{-1}(I_e(W^{-1}J \cap A)^{[p^e]}) = (W^{-1}I_e(W^{-1}J \cap A))^{[p^e]}$. \square

Proposition 9. *Let \widehat{A} denote the completion of A with respect to any prime ideal and W any multiplicatively closed subset of A . Then the following hold:*

- (1) $I_e(J \otimes_A \widehat{A}) = I_e(J) \otimes_A \widehat{A}$, for any ideal $J \subseteq A$;
- (2) $W^{-1}I_e(J) = I_e(W^{-1}J)$.

Proof. (1) Write $\widehat{J} = J \otimes_A \widehat{A}$. Since $I_e(\widehat{J})^{[p^e]} \supseteq \widehat{J}$ using [Ly-Sm, Lemma 6.6] we obtain

$$(I_e(\widehat{J}) \cap A)^{[p^e]} = I_e(\widehat{J})^{[p^e]} \cap A \supseteq \widehat{J} \cap A = J.$$

But $I_e(J)$ is the smallest ideal such that $I_e(J)^{[p^e]} \supseteq J$, so $I_e(\widehat{J}) \cap A \supseteq I_e(J)$ and hence $I_e(\widehat{J}) = (I_e(\widehat{J}) \cap A) \otimes_A \widehat{A} \supseteq I_e(J) \otimes_A \widehat{A}$. On the other hand, $(I_e(J) \otimes_A \widehat{A})^{[p^e]} = I_e(J)^{[p^e]} \otimes_A \widehat{A} \supseteq J \otimes_A \widehat{A}$ and so $I_e(J \otimes_A \widehat{A}) \subseteq I_e(J) \otimes_A \widehat{A}$.

- (2) Since $J \subseteq W^{-1}J \cap A$ then $I_e(J) \subseteq I_e(W^{-1}J \cap A)$ and so $W^{-1}I_e(J) \subseteq W^{-1}I_e(W^{-1}J \cap A)$. By Lemma 8, $W^{-1}I_e(W^{-1}J \cap A) = I_e(W^{-1}J)$ hence $W^{-1}I_e(J) \subseteq I_e(W^{-1}J)$.

For the reverse inclusion it is enough to show that $W^{-1}J \subseteq (W^{-1}I_e(J))^{[p^e]}$ because from this it follows that $I_e(W^{-1}J) \subseteq W^{-1}I_e(J)$ which is what we require. Since $J \subseteq I_e(J)^{[p^e]}$ then $W^{-1}J \subseteq W^{-1}(I_e(J)^{[p^e]}) = (W^{-1}I_e(J))^{[p^e]}$ where in the latter equality we have used Lemma 7. \square

3. THE LOCAL CASE

In this section we give an explicit formula for the HSL-numbers (cf. Definition 2) under some technical hypothesis.

For the moment, assume (R, \mathfrak{m}) is complete and local and let $(-)^{\vee}$ denote the *Matlis dual*, i.e. the functor $\text{Hom}_R(-, E_R)$, where $E_R = E_R(\mathbb{K})$ is the injective hull of the residue field \mathbb{K} of R . We start by recalling the notions of Δ^e -functor and the Ψ^e -functor which have been defined in [K1, Section 3]; let \mathcal{C}^e be the category of Artinian R -modules

with Frobenius maps and \mathcal{D}^e the category of R -linear maps $\alpha_M : M \rightarrow F_R^e(M)$ with M a Noetherian R -module and where a morphism between $M \xrightarrow{\alpha_M} F_R^e(M)$ and $N \xrightarrow{\alpha_N} F_R^e(N)$ is a commutative diagram of R -linear maps:

$$\begin{array}{ccc} M & \xrightarrow{h} & N \\ \alpha_M \downarrow & & \downarrow \alpha_N \\ F_R^e(M) & \xrightarrow{F_R^e(h)} & F_R^e(N). \end{array}$$

We define a functor $\Delta^e : \mathcal{C}^e \rightarrow \mathcal{D}^e$ as follows: given an e^{th} -Frobenius map Θ of the Artinian R -module M , we obtain an R -linear map $\phi : F_*^e(R) \otimes_R M \rightarrow M$ which sends $F_*^e r \otimes m$ to $r\Theta m$. Taking Matlis duals, we obtain the R -linear map

$$M^\vee \rightarrow (F_*^e(R) \otimes_R M)^\vee \cong F_*^e(R) \otimes_R M^\vee$$

where the last isomorphism is the functorial isomorphism described in [L, Lemma 4.1]. This construction can be reversed, yielding a functor $\Psi^e : \mathcal{D}^e \rightarrow \mathcal{C}^e$ such that $\Psi^e \circ \Delta^e$ and $\Delta^e \circ \Psi^e$ can naturally be identified with the identity functor. See [K1, Section 3] for the details of this construction.

From now on let (R, \mathfrak{m}) be a complete, regular and local ring, I an ideal of R and write $S = R/I$. Let d be the dimension of S and suppose S is Cohen-Macaulay with canonical module $\bar{\omega}$. For our purpose, we assume that S is either, generically Gorenstein (i.e. each localisation of S at a prime ideal is Gorenstein), or a domain, or reduced so that $\bar{\omega} \subseteq S$ is an ideal of S , cf. [Hoch, Prop. 14.9]. In which case we can consider the following short exact sequence:

$$0 \rightarrow \bar{\omega} \rightarrow S \rightarrow S/\bar{\omega} \rightarrow 0$$

that induces the long exact sequence

$$\cdots \rightarrow H_{\mathfrak{m}S}^{d-1}(S) \rightarrow H_{\mathfrak{m}S}^{d-1}(S/\bar{\omega}) \rightarrow H_{\mathfrak{m}S}^d(\bar{\omega}) \rightarrow H_{\mathfrak{m}S}^d(S) \rightarrow 0.$$

Since S is Cohen-Macaulay, the above reduces to

$$(1) \quad 0 \rightarrow H_{\mathfrak{m}S}^{d-1}(S/\bar{\omega}) \rightarrow H_{\mathfrak{m}S}^d(\bar{\omega}) \rightarrow H_{\mathfrak{m}S}^d(S) \rightarrow 0.$$

As we noticed in the introduction, a natural Frobenius map acting on S induces a natural Frobenius map acting on $H_{\mathfrak{m}S}^d(S)$. The following result allows us to talk about *the* natural Frobenius map acting on $H_{\mathfrak{m}S}^d(S)$.

Theorem 10 (cf. in Example 3.7 [Ly-Sm]). *Let $\mathcal{F}_e := \mathcal{F}_e(H_{\mathfrak{m}S}^d(S))$ be the R -module consting of all e^{th} -Frobenius maps acting on $H_{\mathfrak{m}S}^d(S)$.*

Then \mathcal{F}_e is generated by one element and the generator corresponds to the natural Frobenius map.

Moreover, we can give an explicit description of the R -module \mathcal{F}_e and consequently of the natural Frobenius which generates it. Our strategy consists of checking that the exact sequence (1) is an exact sequence in \mathcal{C}^e so that we can apply to it the Δ^e -functor defined above.

Firstly, note that the natural Frobenius acting on S induces (by restriction) the natural Frobenius map on $\bar{\omega}$. Hence we also have a natural Frobenius map F acting on $H_{mS}^d(\bar{\omega})$ and the surjection $\alpha: H_{mS}^d(\bar{\omega}) \rightarrow H_{mS}^d(S)$ is compatible with the Frobenius maps (cf. [K1, Section 7]). A Frobenius map is also induced on $H_{mS}^{d-1}(S/\bar{\omega})$ as the restriction of the natural Frobenius map acting on $H_{mS}^d(\bar{\omega})$ to $H_{mS}^{d-1}(S/\bar{\omega})$ and we need to check that this restriction is an endomorphism so that we have that (2) is a short exact sequence in \mathcal{C}^1 . Consider the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{mS}^{d-1}(S/\bar{\omega}) & \longrightarrow & H_{mS}^d(\bar{\omega}) & \xrightarrow{\alpha} & H_{mS}^d(S) \longrightarrow 0 \\ & & \downarrow & & \downarrow F & & \downarrow T \\ 0 & \longrightarrow & H_{mS}^{d-1}(S/\bar{\omega}) & \longrightarrow & H_{mS}^d(\bar{\omega}) & \xrightarrow{\alpha} & H_{mS}^d(S) \longrightarrow 0 \end{array}$$

where T is the natural Frobenius acting on $H_{mS}^d(S)$. To show that $F \text{Ker}(\alpha) \subseteq \text{Ker}(\alpha)$ we use that the surjection $\alpha: H_{mS}^d(\bar{\omega}) \rightarrow H_{mS}^d(S)$ is compatible with the Frobenius maps. Pick $a \in \text{Ker}(\alpha)$ then $F(a) \in \text{Ker}(\alpha)$ because $\alpha(F(a)) = T(\alpha(a)) = T(0) = 0$.

Once we have fixed an isomorphism between $\text{Ann}_{E_R}(I)$ and $H_{mS}^d(\bar{\omega})$ we can then consider the natural Frobenius map, which we call F with an abuse of notation, induced on $\text{Ann}_{E_R}(I)$. By [K1, Prop. 4.1] all Frobenius actions on $\text{Ann}_{E_R}(I)$ are given by the restriction of $uF: E_R \rightarrow E_R$ where $u \in (I^{[p]} : I)$ and F is the natural Frobenius map acting on E_R . Consequently any e^{th} -Frobenius map acting on $\text{Ann}_{E_R}(I)$ is of the form

$$(uF)^e = \underbrace{uF \circ \cdots \circ uF}_{e \text{ times}} = u^{\nu_e} F^e$$

where $\nu_e = 1 + p + \cdots + p^{e-1}$ when $e > 0$ and $\nu_0 = 0$. In the same way we can then prove that (1) is a short exact sequence in \mathcal{C}^e and that the maps are morphisms in \mathcal{C}^e . We can then apply the Δ^e -functor to (1) but before we do that we write such a short exact sequence in a useful way for our computations. We start with the following lemma.

Lemma 11. *Let S and $\bar{\omega}$ be as above then modules $H_{\mathfrak{m}S}^{d-1}(S/\bar{\omega})$ and $\text{Ann}_{E_S}(\bar{\omega})$ are isomorphic.*

Proof. If we identify $H_{\mathfrak{m}S}^d(\bar{\omega})$ with $\text{Ann}_{E_R}(I)$ then $H_{\mathfrak{m}S}^{d-1}(S/\bar{\omega})$ must be of the form $\text{Ann}_{E_S}(J)$ for a certain ideal $J \subseteq S$.

In [L, Lemma 2.16] it has been shown that for a generic ideal $J \subseteq S$ it must be that $\text{Ann}_S(H_{\mathfrak{m}S}^d(S/J)) = (0 : (0 : J))/J$. In our case, $J = \bar{\omega}$ is unmixed and consequently, using Matlis Duality, the result follows. \square

Hence we can rewrite (1) as

$$(2) \quad 0 \rightarrow \text{Ann}_{E_S}(\bar{\omega}) \rightarrow \text{Ann}_{E_R}(I) \rightarrow H_{\mathfrak{m}S}^d(S) \rightarrow 0$$

and therefore we have $H_{\mathfrak{m}S}^d(S) \cong \text{Ann}_{E_R}(I) / \text{Ann}_{E_S}(\bar{\omega})$. An application of the Δ^e -functor to the latter short exact sequence yields the short exact sequence in \mathcal{D}^e :

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bar{\omega}/I & \longrightarrow & R/I & \longrightarrow & R/\omega \longrightarrow 0 \\ & & \downarrow & & \downarrow u^{\nu_e} & & \downarrow \\ 0 & \longrightarrow & \bar{\omega}^{[p^e]}/I^{[p^e]} & \longrightarrow & R/I^{[p^e]} & \longrightarrow & R/\omega^{[p^e]} \longrightarrow 0 \end{array}$$

where the central vertical map is the multiplication by u^{ν_e} . Note that we have used the fact that $F_R(-)$ is exact and R is regular. In order to make the latter a commutative diagram the only possibility is that the other two vertical maps are also the multiplication by u^{ν_e} . Moreover, such a diagram is well defined only if $u \in (I^{[p^e]} : I) \cap (\omega^{[p^e]} : \omega)$ and since $\bar{\omega}$ contains a nonzerodivisor then the kernel of the surjective map $(I^{[p^e]} : I) \cap (\omega^{[p^e]} : \omega) \rightarrow \mathcal{F}_e(H_{\mathfrak{m}S}^d(S))$ which sends $u \mapsto u^{\nu_e}T^e$ is $I^{[p^e]}$. Hence we have the following:

Theorem 12. *The R -module consting of all e^{th} -Frobenius maps acting on $H_{\mathfrak{m}S}^d(S)$ is of the form*

$$\mathcal{F}_e = \frac{(I^{[p^e]} : I) \cap (\omega^{[p^e]} : \omega)}{I^{[p^e]}}$$

where ω is the preimage of $\bar{\omega}$ in R .

Consequently, if Θ is any Frobenius map acting on $H_{\mathfrak{m}S}^d(S)$ then Θ is of the form uT where T is the natural Frobenius map acting on $H_{\mathfrak{m}S}^d(S)$ and $u \in (I^{[p]} : I) \cap (\omega^{[p]} : \omega)$.

We prove now the main result of this section:

Theorem 13. *$\text{HSL}(H_{\mathfrak{m}S}^d(S))$ is the smallest integer e for which*

$$\frac{I_e(u^{\nu_e}\omega)}{I_{e+1}(u^{\nu_{e+1}}\omega)} = 0$$

where ω is the preimage of $\bar{\omega}$ in R and $\nu_e = 1 + p + \cdots + p^{e-1}$ when $e > 0$ and $\nu_0 = 0$.

Proof. We know that any Frobenius map acting on $H_{mS}^d(S)$ is of the form $\Theta = uT$ for some $u \in (I^{[p]} : I) \cap (\omega^{[p]} : \omega)$ and $\Theta^e = u^{\nu_e}T^e$. For all $e \geq 0$ define $M_e = \{x \in H_{mS}^d(S) | T^e(x) = 0\}$ which yields a chain of inclusions of M_e that stabilises by Theorem 1. Moreover, each M_e is a submodule of $H_{mS}^d(S)$ and therefore it is of the form $\frac{\text{Ann}_{E_S} L_e}{\text{Ann}_{E_S} \bar{\omega}}$ for some $L_e \subseteq R$ contained in I . Our goal is to find L_e for all e in such a way that $\frac{\text{Ann}_{E_S} L_e}{\text{Ann}_{E_S} \bar{\omega}}$ is the biggest submodule of $\frac{\text{Ann}_{E_S} I}{\text{Ann}_{E_S} \bar{\omega}}$ such that we have

$$(3) \quad \Theta^e \left(\frac{\text{Ann}_{E_S} L_e}{\text{Ann}_{E_S} \bar{\omega}} \right) \subseteq \frac{\text{Ann}_{E_S} I}{\text{Ann}_{E_S} \bar{\omega}}.$$

Applying Δ^e to the inclusion $\frac{\text{Ann}_{E_S} L_e}{\text{Ann}_{E_S} \bar{\omega}} \hookrightarrow \frac{\text{Ann}_{E_S} I}{\text{Ann}_{E_S} \bar{\omega}}$ we get

$$\begin{array}{ccc} \omega/I & \longrightarrow & \omega/L_e \\ u^{\nu_e} \downarrow & & \downarrow \\ \omega^{[p^e]}/I^{[p^e]} & \longrightarrow & \omega^{[p^e]}/L_e^{[p^e]} \end{array}$$

where the map $\omega/L_e \rightarrow \omega^{[p^e]}/L_e^{[p^e]}$ must be the multiplication by u^{ν_e} because of the surjectivity of the horizontal maps; note that such a map is well defined because $u^{\nu_e}\omega \subseteq \omega^{[p^e]}$, and then $L_e \subseteq \omega$. Moreover $\omega/L_e \rightarrow \omega^{[p^e]}/L_e^{[p^e]}$ must be the zero-map by construction. Hence, $u^{\nu_e}\omega \subseteq L_e^{[p^e]}$, and in order to have the inclusion (3), L_e must be the smallest ideal such that its e^{th} -Frobenius power contains $u^{\nu_e}\omega$ so it must be that $L_e = I_e(u^{\nu_e}\omega)$. \square

Corollary 14. *S is F -injective if and only if $\omega = I_1(u\omega)$.*

Proof. S is F -injective if and only if the index of nilpotency is zero i.e. if and only if $\omega = I_1(u\omega)$. \square

Remark 15. Note that the index of nilpotency gives a measure of 'how far' S is from being F -injective and that Theorem 1 implies that eventually, for a certain integer e , $T^e: H_{mS}^d(S) \rightarrow H_{mS}^d(S)$ is injective.

4. THE NON-LOCAL CASE

In this section let A be a polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ with coefficients in a perfect field of positive characteristic p and $J \subset A$ an ideal of A . Let B be the quotient ring A/J ; if B is Cohen-Macaulay of dimension d then $\bar{\Omega} = \text{Ext}_A^{\dim A - d}(B, A)$ is a global canonical module for

S . Define:

$$(4) \quad \mathcal{U} = \frac{(J^{[p^e]} : J) \cap (\Omega^{[p^e]} : \Omega)}{J^{[p^e]}}$$

where Ω is the preimage of $\bar{\Omega}$ in A . Note that the quantity above is well defined even if $\bar{\Omega}$ is not an ideal of B . Furthermore, \mathcal{U} is finitely generated since it is a module over a Noetherian ring.

Now, for any prime ideal $\mathfrak{p} \subset A$, the A -module consisting of the Frobenius maps on $H_{\mathfrak{p}\hat{B}_{\mathfrak{p}}}^{\dim \hat{B}_{\mathfrak{p}}}(\hat{B}_{\mathfrak{p}})$ is of the form:

$$(5) \quad \frac{(J^{[p^e]}\hat{A}_{\mathfrak{p}} : J\hat{A}_{\mathfrak{p}}) \cap (\Omega^{[p^e]}\hat{A}_{\mathfrak{p}} : \Omega\hat{A}_{\mathfrak{p}})}{J^{[p^e]}\hat{A}_{\mathfrak{p}}}.$$

Moreover

$$\frac{(J^{[p^e]}\hat{A}_{\mathfrak{p}} : J\hat{A}_{\mathfrak{p}}) \cap (\Omega^{[p^e]}\hat{A}_{\mathfrak{p}} : \Omega\hat{A}_{\mathfrak{p}})}{J^{[p^e]}\hat{A}_{\mathfrak{p}}} = \mathcal{U}\hat{A}_{\mathfrak{p}}$$

and $\mathcal{F}_e \cong \mathcal{U}\hat{A}_{\mathfrak{p}}$; consequently $\mathcal{U}\hat{A}_{\mathfrak{p}}$ is generated by one element by Theorem 10.

We prove now the following result for a generic finitely generated A -module.

Theorem 16. *Let M be a finitely generated A -module and let g_1, \dots, g_s be a set of generators for M . If M is locally principal then for each $i = 1, \dots, s$*

$$\mathcal{G}_i = \{\mathfrak{p} \in \text{Spec}(A) \mid M\hat{A}_{\mathfrak{p}} \text{ is generated by the image of } g_i\}$$

is a Zariski open set.

Before we proceed with the proof of Theorem 16 we need the following lemma. Let M be an A -module generated by g_1, \dots, g_s . Let e_1, \dots, e_s be the canonical basis for A^s and define the map

$$\begin{aligned} A^s &\xrightarrow{\varphi} M \\ e_i &\longmapsto g_i. \end{aligned}$$

Such a surjective map extends naturally to an A -linear map $J: A^t \rightarrow A^s$ with $\ker \varphi = \text{Im } J$. Let J_i be the matrix obtained from $J \in \text{Mat}_{s,t}(A)$ by erasing the i^{th} -row. With this notation we have the following:

Lemma 17. *M is generated by g_i if and only if $\text{Im } J_i = A^{s-1}$.*

Proof. Firstly suppose $\text{Im } J_i = A^{s-1}$. We can add to J , columns of $\text{Im } J$ without changing its image so we can assume that J looks like

$$J = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & 1 & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_1 & b_2 & \cdots & b_s \\ \vdots & \vdots & \ddots & \vdots & 0 & \cdots & 1 & 0 \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} & 0 & \cdots & 0 & 1 \end{pmatrix}$$

where $a_{k,l}, b_j \in A$. In this way for every $j = 1, \dots, s$ we have $g_j - b_j g_i = 0$ i.e. g_i generates M . Viceversa if M is generated by g_i then for all $j \neq i$ we can write $g_j = r_j g_i$ i.e. $g_j - r_j g_i = 0$ and this happens if and only if $e_j - r_j e_i = 0$ if and only if $e_j - r_j e_i \in \text{Ker } \varphi = \text{Im } J$. Hence we can assume J to contain the vector whose entries are all zeros but in the i -th and j -th positions where there is 1 and r_j respectively. Consequently J_i contains the $(s-1) \times (s-1)$ identity matrix. \square

Let M be a A -module generated by g_1, \dots, g_s and let W be any multiplicatively closed subset of A . Localise the exact sequence $A^t \rightarrow A^s \rightarrow M \rightarrow 0$ with respect to W obtaining the exact sequence $W^{-1}A^t \rightarrow W^{-1}A^s \rightarrow W^{-1}M \rightarrow 0$. With this notation we have:

Proposition 18. $W^{-1}M$ is generated by $\frac{g_i}{1}$ if and only if $W^{-1}J_i = (W^{-1}A)^{s-1}$

Proof. Apply Lemma 17 to the localised sequence $W^{-1}A^t \rightarrow W^{-1}A^s \rightarrow W^{-1}M \rightarrow 0$. \square

The proof of Theorem 16 follows immediately since the above Proposition is equivalent to saying that the intersecion of W with the ideal of $(s-1) \times (s-1)$ minors of J_i is not trivial.

Proof of Theorem 16. $\mathfrak{p} \in \mathcal{G}_i$ if and only if $\mathfrak{p} \not\supseteq J_i$. \square

From Theorem 16 it follows that if we choose $M = \mathcal{U}$ then for every prime ideal $\mathfrak{p} \in \text{Spec}(A) = \bigcup_i \mathcal{G}_i$ there exists an i such that $\mathfrak{p} \in \mathcal{G}_i$ and the A -module $\widehat{\mathcal{U}}_{\mathfrak{p}}$ is generated by one element which is precisely the image of g_i . Hence, once we have localised and completed B with respect to any prime ideal of A we can use the local theory developed so far. In particular, from Corollary 14 it follows:

Theorem 19. For every prime ideal $\mathfrak{p} \subset A$, $\widehat{B}_{\mathfrak{p}}$ is F -injective if and only if $I_1(u\Omega_{\mathfrak{p}}) = \Omega_{\mathfrak{p}}$.

Remark 20. Since $u \in (\Omega_{\mathfrak{p}}^{[p^e]} : \Omega_{\mathfrak{p}})$ then $u\Omega_{\mathfrak{p}} \subseteq \Omega_{\mathfrak{p}}^{[p^e]}$ and consequently $I_1(u\Omega_{\mathfrak{p}}) \subseteq I_1(\Omega_{\mathfrak{p}}^{[p^e]}) = \Omega_{\mathfrak{p}}$.

Now we prove our main result.

Theorem 21.

$$\mathcal{B}_e = \left\{ \mathfrak{p} \in \operatorname{Spec}(A) \mid \operatorname{HSL} \left(H_{\mathfrak{p}\widehat{B}_{\mathfrak{p}}}^{d(\mathfrak{p})}(\widehat{B}_{\mathfrak{p}}) \right) < e \right\}$$

is a Zariski open set.

Proof. For each $\mathfrak{p} \in \mathcal{G}_i$ consider

$$\frac{u^{\nu_e}}{1} T_{(\mathfrak{p})}^e : H_{\mathfrak{p}B_{\mathfrak{p}}}^{d(\mathfrak{p})}(\widehat{B}_{\mathfrak{p}}) \rightarrow H_{\mathfrak{p}B_{\mathfrak{p}}}^{d(\mathfrak{p})}(\widehat{B}_{\mathfrak{p}})$$

where $T_{(\mathfrak{p})}^e$ is the natural Frobenius acting on $H_{\mathfrak{p}B_{\mathfrak{p}}}^{d(\mathfrak{p})}(\widehat{B}_{\mathfrak{p}})$ and $d(\mathfrak{p}) = \dim \widehat{B}_{\mathfrak{p}}$. Then for all $i = 1, \dots, n$:

$$\begin{aligned} \mathcal{B}_e \cap \mathcal{G}_i &= \left\{ \mathfrak{p} \in \mathcal{G}_i \mid \operatorname{HSL} \left(H_{\mathfrak{p}\widehat{B}_{\mathfrak{p}}}^{d(\mathfrak{p})}(\widehat{B}_{\mathfrak{p}}) \right) < e \right\} = \\ &= \left\{ \mathfrak{p} \in \mathcal{G}_i \mid \frac{I_e(u^{\nu_e}(\widehat{\Omega}_{\mathfrak{p}}))}{I_{e+1}(u^{\nu_{e+1}}(\widehat{\Omega}_{\mathfrak{p}}))} \neq 0 \right\} = \\ &= \left\{ \mathfrak{p} \in \mathcal{G}_i \mid \frac{I_e(u^{\nu_e}(\Omega)) \otimes \widehat{B}_{\mathfrak{p}}}{I_{e+1}(u^{\nu_{e+1}}(\Omega)) \otimes \widehat{B}_{\mathfrak{p}}} \neq 0 \right\} = \\ &= \left\{ \mathfrak{p} \in \mathcal{G}_i \mid \frac{I_e(u^{\nu_e}(\Omega))}{I_{e+1}(u^{\nu_{e+1}}(\Omega))} \otimes \widehat{B}_{\mathfrak{p}} \neq 0 \right\} = \\ &= \operatorname{Supp} \left(\frac{I_e(u^{\nu_e}(\Omega))}{I_{e+1}(u^{\nu_{e+1}}(\Omega))} \right) \cap \mathcal{G}_i. \end{aligned}$$

The latter is a finite set therefore $\mathcal{B}_e \cap \mathcal{G}_i$ is a Zariski open for all i . Since $\cup_i \mathcal{G}_i = \operatorname{Spec}(A)$ then \mathcal{B}_e is Zariski open. \square

Corollary 22. *The index of nilpotency is bounded.*

Corollary 23. *The F -injective locus of the top local cohomology of a quotient of a polynomial ring is open.*

Proof. It is the special case of Theorem 21 when $e = 0$. \square

Note that this argument can be implemented as an algorithm which takes $B = A/J$ as input and produces a set of ideals K_1, \dots, K_t such that, for each $i = 1, \dots, t$, $\mathbb{V}(K_i)$ consists of a prime for which the index of nilpotency is greater than i and $\operatorname{Spec}(B) = \bigcup \mathbb{V}(K_i)$.

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